# On uniqueness techniques for degenerate convection-diffusion problems

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# ON UNIQUENESS TECHNIQUES FOR DEGENERATE CONVECTION-DIFFUSION PROBLEMS

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Abstract. We survey recent developments and give some new results concerning uniqueness of weak and renormalized solutions for degenerate parabolic problems of the form $u_t - \operatorname{div}\,(a_0(\nabla w) + F(w)) = f$, $u \in \beta(w)$ for a maximal monotone graph $\beta$, a Leray-Lions type nonlinearity $a_0$, a continuous convection flux $F$, and an initial condition $u|_{t=0} = u_0$. The main difficulty lies in taking boundary conditions into account. Here we consider Dirichlet or Neumann boundary conditions or the case of the problem in the whole space.

We avoid the degeneracy that could make the problem hyperbolic in some regions; yet our starting point is the notion of entropy solution, notion that underlies the theory of general hyperbolic-parabolic-elliptic problems. Thus, we focus on techniques that are compatible with hyperbolic degeneracy, but here they serve to treat only the "parabolic-elliptic aspects". We revisit the derivation of entropy inequalities inside the domain and up to the boundary; technique of "going to the boundary" in the Kato inequality for comparison of two solutions; uniqueness for renormalized solutions obtained via reduction to weak solutions. On several occasions, the results are achieved thanks to the notion of integral solution coming from the nonlinear semigroup theory.

## 1. Introduction

1.1. **A survey of literature.** Study of degenerate parabolic problems has undergone a considerable progress in the last ten years, thanks to the fundamental paper of J. Carrillo [26] in which the Kruzhkov device of doubling of variables was extended to hyperbolic-parabolic-elliptic problems of the form $j(v) - \operatorname{div}(f(v) + \nabla\varphi(v)) = 0$, and a technique for treating the homogeneous Dirichlet boundary conditions was put forward. In [26], the appropriate notion of entropy solution was established, and this definition (or, sometimes, parts of the uniqueness techniques of [26]) led to many developments; among them, let us mention [2, 3, 5, 4, 6, 8, 9, 10, 11, 12, 13, 18, 19, 24, 25, 27, 29, 30, 34, 37, 38, 39, 41, 42, 45, 46, 47, 48, 52, 53, 57, 58, 59]. Also numerical aspects of the problem were investigated; see, e.g., [7, 32, 33, 35, 40, 49].

The notion of entropy solution (or, as in the present paper, entropy solutions techniques used on weak solutions) was retained by most of the authors; yet, let us mention the version of Bendahmane and Karlsen [18, 19] adapted to anisotropic diffusions, and the fruitful notion of kinetic solution suitable for quasilinear diffusion operators (see in particular Chen and Perthame [31] and the book [56] of Perthame). Derivation of entropy inequalities was revisited by Igbida and Urbano [38] and by the authors [10]. Leray-Lions kind diffusions were considered starting from Carrillo and Wittbold [27]. Triply nonlinear degenerate problems were considered by Ouaro and Touré [53], Ouaro [52] in one space dimension; then by Ammar and Redwane [5], Ammar [2, 3], Andreianov, Bendahmane, Karlsen and Ouaro [8].

As to the treatment of the boundary conditions, it turned out that the techniques of [26] for the homogeneous Dirichlet condition are as much restrictive as

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ingenious (cf. Rouvre and Gagneux [57] for an interpretation of the Carrillo boundary conditions for the case of sufficiently regular solutions). In this note, we survey different techniques and results for treating the boundary (or its absence, for the case $\Omega = \mathbb{R}^d$) within the context of entropy solutions. Notice that in the parabolic-elliptic context and for regular convection flux, one can avoid using entropy solutions and the doubling of variables; then uniqueness results can be obtained for very general nonlinear and dynamical boundary conditions. We refer to Igbida [36], Andreu, Igbida, Mazón and Toledo [14, 15, 16] and references therein.

Further, many of the works cited above were devoted to renormalized solutions, starting from Carrillo and Wittbold [27]. General existence and uniqueness techniques for renormalized solutions of convection-diffusion problems are by now well established; but they are quite heavy, therefore arguments allowing to simplify the proofs are of interest. For proving existence or renormalized solutions, a key idea is to use bi-monotone approximations of Ammar and Wittbold [6]; this ensures strong compactness through monotonicity (unfortunately, this technique cannot be applied for measure data, but only to $L^1$ data). In the context of degenerate problems, compactness is enforced through penalization by a strictly monotone absorption term (see Sbihi and Wittbold [58], Zimmermann [60]). For uniqueness, the idea of reduction to $L^1$ contraction for weak solutions for an auxiliary problem was proposed by Igbida and Wittbold (see [37]; see also [11]); in this note, we will revisit and generalize this idea.

Nonlinear semigroup techniques were used in [26], and in many subsequent papers. In this approach, one first studies in detail the associated stationary (degenerate elliptic) problem, and then uses the Crandall-Liggett theorem and the related notions of mild and integral solutions (see Bénilan [20], Bénilan, Crandall and Pazy [22], Bénilan and Wittbold [21]). Whereas a direct study of solutions for the degenerate parabolic problem remains possible in many cases, one truly simplifies the existence and/or uniqueness proofs using powerful abstract tools of [20, 22]. The direct methods remain necessary, e.g., for problems with explicit dependence on time variable $t$. In this note, we highlight the applications for which a direct study of uniqueness for the evolution problem appears as problematic or highly technical, and the use of semigroup techniques offers fair advantages (cf. [21]). The main idea is the following: one needs to compare two solutions to the evolution problem, and it turns out that it is simpler to compare a solution to the evolution problem with a (somewhat more regular) solution to the associated stationary problem. Then it is possible to deduce that a solution to the evolution problem is an integral solution; and then refer to the uniqueness of integral solutions, granted by the general theory of nonlinear semigroups. Detailed examples are given in Andreu, Igbida, Mazón, Toledo [14, 16], Andreianov and Bouhsiss [9] (cf. Section 3.3.2) and in Section 3.

1.2. **Stefan-type degenerate convection-diffusion equations.** In the present contribution, we will survey several aspects of the aforementioned works, mostly related to the works of the authors. Unless the contrary is stated, we are restricted to the "weakly degenerate" convection-diffusion problems of parabolic-elliptic type; for these problems, weak and entropy solutions are equivalent. More precisely, we consider the PDEs under the following general form :

$$j(v)_t - \operatorname{div}\, a(w,\nabla w) = f, \quad w = \varphi(v) \quad \text{in } Q = (0,T)\times\Omega \subset \mathbb{R}^+\times\mathbb{R}^d, \tag{1}$$

sometimes referred to as Stefan type problems. Here $j,\varphi$ are two continuous nondecreasing functions on $\mathbb{R}$, normalized by $j(0) = \varphi(0) = 0$; and $a : \mathbb{R}\times\mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a continuous function satisfying generalized Leray-Lions conditions. As it was pointed out in [26, 10], considering such nonlinearities $j$ and $\varphi$ is equivalent to considering a maximal monotone graph $\beta$ on $\mathbb{R}$ with $0 \in \beta(0)$; the corresponding problem writes $u_t - \operatorname{div}\, a(w,\nabla w) = f$ with $u \in \beta(w)$ (setting $j = (I+\beta^{-1})^{-1}$, $\varphi = (I+\beta)^{-1}$ and $v := u + w$, we get back to problem (1)). For our purposes, the representation of the problem in terms of $j,\varphi$ is somewhat more convenient. Finally, $f$ represents a source term. In the most general setting, $f$ could be a Radon measure. Within the framework of weak solutions (respectively, of renormalized solutions), we will assume that $f \in L^p((0,T)\times\Omega)$ (resp., $f \in L^1((0,T)\times\Omega)$).

For references on motivations, results and techniques on the Stefan type equations (1) complementary to those discussed in this paper, we refer to [14, 15, 16, 24, 58, 37] and the references given therein.

We will consider the nonlinear diffusion-convection operators corresponding to

$$a(r,\xi) = S(r)a_0(\xi) + F(r) \tag{2}$$

with $a_0 \colon \mathbb{R}^d \longrightarrow \mathbb{R}^d,\ F \colon \mathbb{R} \longrightarrow \mathbb{R}^d$ continuous, satisfying the following assumptions:

$$a_0(\xi)\cdot\xi \geq \frac{1}{C}|\xi|^p, \tag{3}$$

$$(a_0(\xi) - a_o(\eta))\cdot(\xi-\eta) \geq 0, \tag{4}$$

$$|a_0(\xi)|^{p'} \leq C(1+|\xi|^p), \tag{5}$$

$$|F(r)|^{p'} \leq C(1+|r|^p), \tag{6}$$

for some $p \in (1,+\infty)$ and some $C>0$; here $p' = \frac{p}{p-1}$, and $r\in\mathbb{R}$, $\xi,\eta\in\mathbb{R}^d$ are arbitrary. In some of the works we cite, growth assumptions on $F$ different from (6) are considered.

For the nonlinearity $S$, we assume either that

$$0 \leq S \leq C, \qquad S \in W^{1,\infty}(\mathbb{R}) \tag{7}$$

(needed for the study of renormalized solutions) or that $S$ is continuous and

$$\frac{1}{C} \leq S(r) \leq C, \tag{8}$$

(here we develop a new version of the doubling of variables device, see Section 3.2). The case without $S$ was studied in most of the works on the subject; when $a_0$ is homogeneous of degree $p$ (this is the case for the well-known $p$-laplacian), by a suitable change of the nonlinearities $\varphi$ and $F$ we can reduce (2) to the case $S\equiv 1$. Our interest in introducing factor $S$ satisfying (7) becomes apparent in Section 6.

Let us stress that because the convection flux $F$ is assumed merely continuous, uniqueness techniques for (1),(2) are those of entropy solutions (see, e.g., the discussion in [9]). The lack of regularity of $F$ is the only reason why the doubling of variables in space can be needed for the Stefan-type problems (1) (the doubling of variables in time, see Otto [51], Blanchard and Porretta [24], does not interfere with different boundary conditions; moreover, it can be avoided thanks to the nonlinear semigroup techniques, see Bénilan and Wittbold [21] and Section 5). Let us also mention that for diffusion-convection operators under the general form $-\operatorname{div} a(t,x;w,\nabla w)$, the explicit dependence in $x$ is a major obstacle to apply the doubling of variables technique (except for the case treated by Vallet in [59]); some results for this case were obtained by Blanchard and Porretta in [24] and by Zimmermann [60] under regularity assumptions on $F$.

Because most of the difficulties treated in this paper only come from the lack of regularity of the convection flux $F$, the difficulties may seem artificial. Yet the Stefan type problems with continuous $F$ serve as a playground for the wide class of practically important hyperbolic-parabolic-elliptic problems (see in particular [26, 47, 48, 59, 7]); for these problems, entropy inequalities and the doubling of variables remain the essential technique. It is an open question how to transfer to this context the techniques of [9, 12] or those of [11] recalled in this paper; some work in this direction is in progress.

1.3. **Brief outline.** The reader is assumed to be acquainted with the definitions and techniques of the papers [43] by Kruzhkov and [26] by Carrillo. The material is ordered in different Sections as follows. In Section 2 we define different notions of solution and fix our framework. Section 3 is devoted to techniques for getting the so-called Kato inequalities for comparison of two solutions; more precisely, we compare a solution to a stationary solution. We give the argument based upon the test functions of Blanchard and Porretta [24] (cf. [16]) and combine them with the doubling of space variables. In Section 4 we discuss the extension of local Kato inequalities up to the boundary or to the whole space $\mathbb{R}^N$. In Section 5 we discuss the use of nonlinear semigroup techniques for proving uniqueness. Finally, in Section 6 we describe the hint that allows to study uniqueness of renormalized solutions by reduction to weak solutions.

Many references are given at the end of the paper; this list is far from being exhaustive, in particular further relevant references can be found in the works cited.

## 2. Assumptions on the data and definition of solutions

Let $T>0$ be fixed. Except in Section 4.1 where $\Omega=\mathbb{R}^d$, we consider bounded domain with Lipschitz boundary $\Omega\subset\mathbb{R}^d$. Write $Q=(0,T)\times\Omega$ (some of the methods we survey allow for a less regular domain, see e.g. [10, Sect.4] and [11]). In order to embed both Dirichlet and Neumann boundary conditions (BC, for short) into one single formulation, assume that either $\partial\Omega=\Gamma_D$ or $\partial\Omega=\Gamma_N$. See [12] for results on mixed boundary conditions. We consider the following boundary conditions:

$$\text{if } \Gamma_N=\emptyset, \quad w|_{(0,T)\times\Gamma_D}=g, \tag{9}$$

with $g\in L^p(0,T;W^{1-1/p,p}(\Gamma_D))$ (we identify $g$ with an $L^p(0,T;W^{1,p}(\Omega))$ function);

$$\text{if } \Gamma_D=\emptyset, \quad a(w,\nabla w)\cdot n|_{(0,T)\times\Gamma_N}=s \tag{10}$$

with $s\in L^1((0,T)\times\Gamma_N)\cap L^{p'}(0,T;W^{1/p'-1,p'}(\Gamma_N))$; here $n$ is the outer unit normal vector to $\Gamma_N$.

Condition (9) can be rigorously interpreted in terms of strong boundary traces, or, equivalently, as $w-g\in L^p(0,T;W_0^{1,p}(\Omega))$. Condition (10) can be rigorously interpreted in terms of the weak normal trace (in the $L^{p'}(0,T;W^{1/p'-1,p'}(\partial\Omega))$ sense) of the divergence-measure field $(j(u),a(w,\nabla w))$ on $(0,T)\times\Gamma_N$ (see [28]). For the sake of simplicity, assume that $j$ is surjective in the case of Neumann BC:

$$\text{if } \Gamma_D=\emptyset, \quad j(\mathbb{R})=\mathbb{R}. \tag{11}$$

We refer to the works of Andreu, Igbida, Mazón and Toledo [14, 15, 16] for precise solvability assumptions for the case of Neumann boundary conditions and general nonlinear dynamical boundary conditions for Stefan type problems.

Further, consider a measurable $\overline{\mathbb{R}}$-valued function $v_0$ on $\Omega$ such that $j(v_0)=j_0$, and put the initial datum

$$j(v)|_{t=0}=j_0 \quad \text{on } \Omega, \qquad j_0\in L^1(\Omega,\overline{j(\mathbb{R})}); \tag{12}$$

recall that $f$ is the source term in (1) and assume

$$\int_0^{v_0}\varphi(r)\,dj(r)\in L^1(\Omega) \qquad \text{and} \qquad f\in L^p(Q) \tag{13}$$

(for the case of weak solutions) or assume $j_0\in L^1(\Omega)$ and $f\in L^1(Q)$ (for the case of renormalized solutions).

Notice that the assumptions we put on $g,s$ and $j_0,f$ are compatible with the framework of weak solutions (also called variational solutions or energy solutions), in the sense that existence of a weak solution can be shown, e.g., with the methods of Alt and Luckhaus [1] and the penalization and comparison techniques of Ammar and Wittbold [6]; the assumptions on $g,s$ can be relaxed if renormalized solutions are considered. We refer to [59, 24, 37, 60] (for Dirichlet BC), to [9, 16] (for Neumann BC) and to [12] (for mixed BC) for an exposition of different existence techniques and results, under adequate assumptions on $\beta$ and $a$.

Let us first define the notion of local weak solution, before including the boundary conditions into the formulation.

**Definition 2.1.** *A measurable function $v$ on $(0,T)\times\Omega$ such that $\int_0^v r\,dj(r)\in L^1(Q)$ and $w:=\varphi(v)\in L^p(0,T;W^{1,p}(\Omega))$ is a local weak solution of (1), (2) with initial datum (12) if*

$$-\iint_Q j(v)\xi_t-\int_\Omega j_0\xi(0,\cdot)+\iint_Q a(w,\nabla w)\cdot\nabla\xi=\iint_Q f\xi \tag{14}$$

*for all $\xi\in\mathcal{D}([0,T)\times\Omega)$.*

Notice that due to the Sobolev embeddings, under the growth assumptions (8),(6) and also thanks to (11) and the integrability assumption on $j(v)$, the term $a(w,\nabla w)=F(w)+S(w)a_0(\nabla w)$ belongs to $L^{p'}(Q)$. Thus all terms in (14) make sense. Further, by approximation we can take in (14) a test function $\xi\in L^p(0,T;W_0^{1,p}(\Omega))\cap L^\infty(Q)$ with $\xi_t\in L^\infty(Q)$ and $\xi(T,\cdot)=0$, in which case we use the $W^{-1,p'}-W_0^{1,p}$ duality product $<\cdot,\cdot>$ in order to state the identification

$$-\iint_Q j(v)\xi_t-\int_\Omega j_0\xi(0,\cdot)=\int_0^T <j(v)_t,\xi>; \tag{15}$$

this representation is needed because it allows to use the celebrated Alt-Luckhaus chain rule argument, see [1, 27] (cf. the definition of a renormalized solution below). Finally, notice that (14) remains unchanged if we add in the left-hand side the term $-\iint_Q D\cdot\nabla\xi$ with $D\in\mathbb{R}^d$ a constant vector; indeed, this additional term is zero.

**Definition 2.2.** *A local weak solution of (1), (2) with initial datum (12) solves the Dirichlet problem with datum $g$ if, in addition, (9) holds in the sense of traces. A local weak solution solves the Neumann problem with datum $s$ if, in addition, (10) holds in the sense of weak normal traces.*

In the Neumann case, we get (14) with $\xi\in L^p(0,T;W^{1,p}(\Omega))$ and the additional boundary term $-\int_0^T\int_{\Gamma_N}s\xi$ in the right-hand side. Indeed, approximating $\xi$ in the appropriate sense by $L^p(0,T;W_0^{1,p}(\Omega))$ functions (e.g., multiplying $\xi$ by the cut-off functions $\xi_h^0:=\min\{1,\frac{1}{h}\mathrm{dist}\,(x,\partial\Omega)\}$), according to the definition of weak normal trace by Chen and Frid [28]) we generate the boundary term coming from (10).

Further, for $k>0$ and $r\in\overline{\mathbb{R}}$, introduce the truncation function at the level $k$ by $T_k(r)=\mathrm{sign}\,r\,\min\{|r|,k\}$. Let us define the renormalized solutions for the case of Dirichlet data; note that all terms in the definition make sense (see, e.g., [37]).

**Definition 2.3.** *Let $a(r,\xi)=a_0(\xi)+F(r)$, under the assumptions (3)–(5) for $a_0$ and the mere continuity assumption for $F$.*

*A measurable $\overline{\mathbb{R}}$-valued function $v$ on $Q$ is a local renormalized solution of (1) with initial datum (12) if for all $k>0$, $T_k(w)\in L^p(0,T;W^{1,p}(\Omega))$ (here $w:=\varphi(v)$) and*

*(i) for any compactly supported $S\in W^{1,\infty}(\mathbb{R})$, the $\mathcal{D}'$ derivative $\Big(\int_0^v S(\varphi(z))\,dj(z)\Big)_t$ is identified with $\chi_S\in L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)$ by the relation*

$$\int_0^T<\chi_S,\xi>=-\iint_Q\Big(\int_0^v S(\varphi(z))\,dj(z)\Big)\xi_t-\int_\Omega\Big(\int_0^{v_0(x)}S(\varphi(z))\,dj(z)\Big)\xi(0,x)$$

*for all $\xi\in L^p(0,T;W_0^{1,p}(\Omega))\cap L^\infty(Q)$ such that $\xi_t\in L^\infty(Q)$ and $\xi(T,\cdot)=0$; and for all test function $\xi\in L^p(0,T;W_0^{1,p}(\Omega))\cap L^\infty(Q)$ the renormalized equation holds:*

$$\int_0^T<\chi_S,\xi>+\iint_Q a(w,\nabla w)\cdot\nabla(S(w)\xi)=\iint_Q fS(w)\xi;$$

*(ii) the following integrability constraint holds:*

$$\iint_{\{(t,x)\in Q\,|\,M-1\le|w(t,x)|\le M\}}|\nabla w|^p\to 0\quad\text{as } M\to\infty.$$

*A local renormalized solution of (1) with initial datum (12) solves the Dirichlet problem with datum $g$ if, in addition, for all $k>0$ one has $T_k(w-g)=0$ on $(0,T)\times\partial\Omega$ in the sense of traces.*

Let us point out that the above constraint (ii) is slightly different from what is usually required in the definition of a renormalized solution: indeed, in view of the growth and coercivity conditions on $a_0$, what we require is the convergence to zero in $L^1(Q)$ of the non-negative functions $R_M := a_0(\nabla w)\cdot\nabla w 1\!\!1_{\{M-1\le|w(t,x)|\le M\}}$ as $M\to\infty$, while the usual form of the constraint (as imposed, e.g., in [27, 6, 60]) is, $\iint_Q a(w,\nabla w)\cdot\nabla w 1\!\!1_{\{M-1\le|w(t,x)|\le M\}}\to 0$. These two conditions are equivalent in the case $g\equiv 0$, thanks to the chain rule and integration-by-parts arguments for the term $\iint_Q F(w)\cdot\nabla w 1\!\!1_{\{M-1\le|w(t,x)|\le M\}}$. In the general case, in order to get existence of renormalized solutions to the Dirichlet problem according to Definition 2.3, the conditions $\widetilde{R}^{\pm}_M(t)\to 0$ as $M\to\infty$ should be imposed, where $\widetilde{R}^{\pm}_M(t) := \int_{\partial\Omega}\big(\int_{M-1}^{t_M(g(t,\cdot)^{\pm})} F(s)\,ds\big)\cdot n$ and $t_M(z) := \min\{M,\max\{z,M-1\}\}$.

**Remark 2.4** (An Erratum)**.** *We should mention in passing that in Andreianov and Igbida* [11]*, Definition 7.1 of renormalized solutions is wrong; for a formulation leading to the uniqueness result of* [11, Theorem 7.2]*, one should take (i),(ii) of the above Definition 2.3.*

Whenever we speak of uniqueness of weak ( respectively, renormalized) solutions, we actually mean the uniqueness of $j(v)$ such that $v$ is a weak (resp., renormalized) solution of the problem.

Although we are concerned with uniqueness results for weak or renormalized solutions, the essential tool of our study are the entropy inequalities. Introduce $\mathrm{sign}^{\pm}(r) = \pm\mathrm{sign}\,(r^{\pm})$ and the associated non-decreasing Lipschitz approximations

$$H^{\pm}_{\varepsilon}(r) = \pm\min\{\tfrac{r^{\pm}}{\varepsilon},1\}$$

of $\mathrm{sign}^{\pm}(r)$, for $\varepsilon>0$. Then, according to Carrillo [26], for the case $S\equiv 1$ we can get the inequalities

$$\begin{aligned}
&\forall\, D\in\mathbb{R}^d\ \ \forall\, k\in\mathbb{R}\ \text{ and } \kappa=\varphi(k),\ \ \text{there holds}\\
&\big((j(v)-j(k))^{\pm}\big)_t \mp \mathrm{div}\ \mathrm{sign}^{\pm}(w-\kappa)\big((F(w)-F(\kappa)) + (a_0(\nabla w)-D)\big)\\
&\quad\le\ \pm f\mathrm{sign}^{\pm}(j(v)-j(k)) - \lim_{\varepsilon\to 0}\Big\{\big(H^{\pm}_{\varepsilon}\big)'(w-\kappa)(a_0(\nabla w)-D)\cdot\nabla w\Big\}\\
&\quad\text{with}\quad (j(v)-j(k))^{\pm}\big|_{t=0}\le (j_0-j(k))^{\pm}\\
&\hspace{20em}\text{in } \mathcal{D}'([0,T)\times\Omega)
\end{aligned}\tag{16}$$

for local weak solutions of (1), (2). In fact, the $\lim_{\varepsilon\to 0}$ term in the inequality (16) exists and has the sense of a measure on $Q$. Using the idea of [24] we will get slightly different entropy inequalities which still can be used to get Kato inequalities.

Finally, notice that in several occasions we will need the stationary problem associated with (1), namely,

$$j(v) - \mathrm{div}\ a(w,\nabla w) = h,\quad w=\varphi(v)\quad\text{in }\Omega,\tag{17}$$

with Dirichlet or Neumann boundary conditions; the notion of a weak solution is a straightforward simplification of Definitions 2.1, 2.2 (one can consider it as a stationary solution to (1) with the source $f := h-j(v)$). We will also need the notion of integral solution (see Bénilan [20], Bénilan, Crandall and Pazy [22], Barthélemy and Bénilan [17]) for the abstract evolution problem associated with (1); for these techniques, we assume that the boundary conditions $g$ or $s$ are time-independent.

**Definition 2.5.** *A function $u\in L^1(Q)$ is an integral solution of equation $u\in\beta(w)$, $u_t-\mathrm{div}\,a(w,\nabla w)=f$ a.e. on $Q$ with initial datum* (12) *and BC $g=g(x)$ in* (9) *(resp., with BC $s=s(x)$ in* (10)*) if for all $(\hat v,\hat f)$ such that $\hat v$ is a weak solution of* (17) *with source $h=j(\hat v)+\hat f$ and with Dirichlet BC $g(x)$ (resp., with Neumann*

*BC $s(x)$) there holds*

$$(18)\quad \frac{d}{dt}\|u(t)-j(\hat v)\|_{L^1(\Omega)} \le \Big[u(t)-j(\hat v), f(t)-\hat f\Big]_{L^1(\Omega)} := \int_\Omega \mathrm{sign}\,(u(t)-j(\hat v))(f(t)-\hat f) + \int_\Omega 1\!\!1_{u(t)=j(\hat v)}|f(t)-\hat f|$$

*in $\mathcal{D}'((0,T))$, and $u(t)\to j_0$ in $L^1(\Omega)$ as $t\to 0$ (excepting a set of measure zero).*

It should be stressed that the choice of $\hat f$ in the above definition may vary; it should run over a dense subset of $L^1(\Omega)$. Further, the requirement that $u \in C([0,T];L^1(\Omega))$ (included in the definition of [20, 22]) is in fact not needed for the proof of the key uniqueness result (cf. [17, 21]); the time continuity follows *a posteriori* as a consequence of identification of integral and mild solutions.

**Remark 2.6.** *Notice that time-dependent Neumann boundary conditions $s$ can be taken into account, if one works on the space $L^1(\Omega)\times L^1(\partial\Omega)$; see Igbida [36], Andreu, Igbida, Mazón and Toledo [14, 16] for the details of the construction.*

*As to the time-dependent Dirichlet boundary conditions $g$, piecewise constant in $t$ conditions can be taken into account directly, by subdividing the time interval. To our knowledge, uniqueness for general time-dependent Dirichlet conditions cannot be studied with the techniques of [20, 22, 17, 21].*

## 3. Getting Kato inequalities

The goal of this section is to deduce the so-called local (away from $\partial\Omega$) Kato inequalities: for $v,\hat v$ weak solutions of (1),(2) with respective data $v_0, f$ and $\hat v_0, \hat f$,

$$(19)\quad -\iint_Q (j(v)-j(\hat v))^\pm\,\xi_t + \iint_Q \mathrm{sign}^\pm(w-\hat w)\big(a(w,\nabla w)-a(\hat w,\nabla\hat w)\big)\cdot\nabla\xi \le \int_\Omega (j_0-\hat j_0)^\pm\,\xi(0,\cdot) + \Big[j(v)-j(\hat v),(f-\hat f)\,\xi\Big]_{L^1(Q)}$$

for all $\xi\in\mathcal{D}([0,T)\times\Omega)$, $\xi\ge 0$. For a merely continuous convection flux $F$ in (2), entropy inequlities and the doubling of variables techniques are needed to deduce the Kato inequalities (19).

Entropy inequalities for (1) (as a particular case) were derived by Carrillo from the weak formulation with the help of the test functions $H^\pm_\varepsilon(w-\kappa)\xi$, as $\varepsilon\to 0$. This leads to inequalities (16) with $k=\min\{\varphi^{-1}(\kappa)\}$ and with $k=\max\{\varphi^{-1}(\kappa)\}$; then a "passage inside the flat regions" is needed in order to recover (16) with $k$ in the interior of the interval $\varphi^{-1}(\kappa)$. This technique of [26] was further developed in [38]; a non-restrictive in practice technical assumption on $\varphi$ was required.

An alternative approach, that we further develop in this note, was proposed by Blanchard and Porretta in [24]. The argument is quite quick for the stationary problem (17). One takes $H^\pm_\varepsilon(w-\kappa+\varepsilon\pi)\xi$ for the test function, where $\pi\in\mathcal{D}(\Omega)$ is a regularization of $\mathrm{sign}^\pm(j(v)-j(k))$. The key observations are: the term in $\nabla H^\pm_\varepsilon(w-\kappa+\varepsilon\pi)$ containing $\nabla\pi$ is the integral of an $L^1$ function independent of $\varepsilon$ over a set of vanishing measure, as $\varepsilon\to 0$, thus this term is harmless; and

$$\lim_{\varepsilon\to 0} H^\pm_\varepsilon(w-\kappa+\varepsilon\pi) = \mathrm{sign}^\pm(w-\varphi(k)) + \pi 1\!\!1_{w=\varphi(k)} =: H^\pm(v,k;\pi),$$

where the limit is in the a.e. sense with a uniform $L^\infty$ bound. We have the convergence $(j(v)-j(k))H^\pm(v,k;\pi)\to(j(v)-j(k))^\pm$ as $\pi\to\mathrm{sign}^\pm(j(v)-j(k))$.

In relation with the notion of integral solution (see Definition 2.5), a careful refinement of the Blanchard-Porretta technique was proposed by Andreu, Igbida, Mazón and Toledo in [16]; the authors compare one solution of the evolution problem to one solution of the stationary problem. In Section 3.1 we give another version of the argument, using the doubling of the space variable.

### 3.1. **Kato inequalities for (1),(2) with Lipschitz continuous $S$.**

In this section we assume that $S$ is of the kind (7); this is the framework needed later in Section 6. First, introduce the following definitions.

Recall that $H^{\pm}_\varepsilon(r)=\pm\min\{\frac{r^\pm}{\varepsilon},1\}$. For $\varepsilon>0$, $z,k\in\mathbb{R}$, $\pi\in[-1,1]$, set

$$\begin{aligned} J^\sharp_\varepsilon(z,k;\pi) &:= \int_k^z H^+_\varepsilon(\varphi(r)-\varphi(k)+\varepsilon\pi)\,dj(r),\\ J^\flat_\varepsilon(k,z;\pi) &:= -\int_k^z H^+_\varepsilon(\varphi(k)-\varphi(r)+\varepsilon\pi)\,dj(r),\end{aligned}$$

$$\begin{aligned} H(z,k;\pi) &:= \operatorname{sign}^+(\varphi(z)-\varphi(k))+\pi 1\!\!1_{\varphi(z)=\varphi(k)}\\ J^\sharp(v,k;\pi) &:= \int_k^z \left[\operatorname{sign}^+(\varphi(r)-\varphi(k))+\pi 1\!\!1_{\varphi(r)=\varphi(k)}\right]dj(r)\\ J^\flat(v,k;\pi) &:= \int_z^k \left[\operatorname{sign}^+(\varphi(k)-\varphi(r))+\pi 1\!\!1_{\varphi(r)=\varphi(k)}\right]dj(r).\end{aligned}$$

Note the following lemma (further, analogous definitions for $H^-_\varepsilon$, $\operatorname{sign}^-$ in the place of $H^+_\varepsilon$, $\operatorname{sign}^+$ yield analogous results); the proof is straightforward.

**Lemma 3.1.** *There holds*

$$\lim_{\varepsilon\to0} H^+_\varepsilon(\varphi(z)-\varphi(k)+\varepsilon\pi)=H(z,k;\pi)$$

$$\lim_{\varepsilon\to0} J^\sharp_\varepsilon(z,k;\pi)=J^\sharp(v,k;\pi),\qquad \lim_{\varepsilon\to0} J^\flat_\varepsilon(k,z;\pi)=J^\flat(v,k;\pi).$$

*Moreover,* $0\le H(r_1,r_2;\pi)\le1$, *and the following properties hold:*

$$J^{\sharp,\flat}(r_1,r_2;\pi)\to(j(r_1)-j(r_2))^+ \ \text{ as } \ \pi\to\operatorname{sign}^+(j(r_1)-j(r_2)), \tag{20}$$

$$\left|J^{\sharp,\flat}(r_1,r_2;\pi)\right|\le|j(r_1)-j(r_2)| \ \text{ uniformly in } \pi, \tag{21}$$

$$J^{\sharp,\flat}\big(r_1,r_2;\operatorname{sign}^+(j(r_1)-j(r_2))\big)\ge J^{\sharp,\flat}\big(r_1,r_2;\pi\big) \ \text{ for all } \pi. \tag{22}$$

Now, following [43, 26] we double the space variable. The doubling of the time variable (which is unnecessary because $\hat v$ is stationary) is also carried out up to a certain point; then, facing technical difficulties we recall that $\hat v$ is time-independent, and therefore we are able to conclude the proof rather quickly (cf. the original argument in [24] where the authors manage to compare two time-dependent solutions).

Consider $\xi\in\mathcal{D}([0,T)^2\times\Omega^2)$ and an auxiliary $[0,1]$-valued function $\pi\in\mathcal{D}(\Omega\times\Omega)$. For fixed $(s,y)\in Q$, we take $H_\varepsilon(w(t,x)-\hat w(s,y)+\varepsilon\pi(x,y))\xi(t,s,x,y)$ for the test function in the weak formulation of (1) (recall (15)). Write $k=\hat v(s,y)$. Pick $D:=a(\hat w(s,y),\nabla_y\hat w(s,y))$ and integrate in $(s,y)\in Q$. Using the Alt-Luckhaus chain rule as in Definition 2.3, we write

$$\int_0^T < j(v)_t, H^+_\varepsilon(w-\varphi(k)+\varepsilon\pi)\xi > = -\iint_Q J^\sharp_\varepsilon(v,k;\pi)\xi_t-\int_\Omega J^\sharp_\varepsilon(v,k;\pi)\xi(0,\cdot), \tag{23}$$

where the integration is in $t$ and in $x$. Using Lemma 3.1, we easily pass to the limit as $\varepsilon\to0$ in the right-hand side of (23). The right-hand side term in (14) yields the limit

$$\lim_{\varepsilon\to0}\iint_Q f\,H_\varepsilon(w-\varphi(k)+\varepsilon\pi))\,\xi=\iint_Q f\,H(v,k;\pi)\,\xi.$$

In the diffusion terms, we add the zero term $\iint_Q D\cdot\nabla\xi$ in the formulation (14), then we get

$$\begin{aligned}\iint_Q &\big(a(w,\nabla w)-D\big)\cdot\big(\nabla_x H_\varepsilon(w(t,x)-\varphi(k)+\varepsilon\pi(x,y))\big)\,\xi\\ &+\iint_Q\big(a(w,\nabla w)-D\big)\cdot\big(\nabla_x\xi\big)\,H_\varepsilon(w(t,x)-\varphi(k)+\varepsilon\pi(x,y));\end{aligned} \tag{24}$$

we pass to the limit in the second term using Lemma 3.1. Notice that all the above limits can be interchanged with the integration in $(s,y)\in Q$.

As to the first term in (24), integrated in $(s,y)$, it yields

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\iint\!\!\iint_{[0<w(t,x)-\hat{w}(s,y)+\varepsilon\pi(x,y)<\varepsilon]}\big(a(w,\nabla w)-D\big)\cdot\nabla_x w\,\xi \\ +\lim_{\varepsilon\to 0}\iint\!\!\iint_{[0<w(t,x)-\hat{w}(s,y)+\varepsilon\pi(x,y)<\varepsilon]}\big(a(w,\nabla w)-D\big)\cdot\nabla_x\pi\,\xi; \tag{25}$$

here we keep the first term, and we notice that the second term amounts to

$$\iint\!\!\iint_{[w(t,x)=\hat{w}(s,y)]}\big(a(w,\nabla w)-a(\hat{w},\nabla\hat{w})\big)\cdot\nabla_x\pi\,\xi.$$

The latter integral is zero because $a(w,\nabla_x w)-a(\hat{w},\nabla_y\hat{w})=0$ a.e. on the set $\big[\,w(t,x)-\hat{w}(s,y)=0\,\big]$, by (2) and by the chain rule property applied to the $L^p((0,T)^2,W^{1,p}(\Omega^2))$ function $(t,s,x,y)\mapsto w(t,x)-\hat{w}(s,y)$. Finally, we combine the terms of the above calculation into one single integral identity. Notice that the first limit in (25) does exist, due to this identity.

In the same way, we take the second weak solution $\hat{u}$ in variables $(s,y)$ corresponding to the data $\hat{v}_0,\hat{f}$. We fix $(t,x)\in Q$ and apply the test function $H_\varepsilon^+(\varphi(k)-\hat{w}(s,y)+\varepsilon\pi(x,y))$ with $k:=v(t,x)$. With analogous calculations, using $J^\flat(k,\hat{v};\pi)$ in the place of $J^\sharp(v,k;\pi)$, we transform the integral identity, pass to the limit as $\varepsilon\to 0$, pick $D:=a(w(t,x),\nabla_x w(t,x))$ and integrate in $(t,x)\in Q$. Subtracting the two obtained identities, we eventually get

$$-\iint_Q\iint_Q\Big[J^\sharp(v,\hat{v};\pi)\,\xi_t+J^\flat(v,\hat{v};\pi)\,\xi_s\Big] \\ -\iint_Q\int_\Omega J^\sharp(v_0,\hat{v};\pi)\,\xi(0,s)-\int_\Omega\iint_Q J^\flat(v,\hat{v}_0;\pi)\,\xi(t,0) \\ +\iint_Q\iint_Q H(v,\hat{v},\pi)\big(a(w,\nabla w)-a(\hat{w},\nabla\hat{w})\big)\cdot(\nabla_x+\nabla_y)\xi \\ +\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\iint\!\!\iint_{[0<w(t,x)-\hat{w}(s,y)+\varepsilon\pi(x,y)<\varepsilon]}\big(a(w,\nabla w)-a(\hat{w},\nabla\hat{w})\big)\cdot\big(\nabla w-\nabla\hat{w}\big)\,\xi \\ =\iint_Q\iint_Q H(v,\hat{v},\pi)\,(f-\hat{f}) \tag{26}$$

for all $\xi\in\mathcal{D}([0,T)^2\times\Omega^2)$, $\pi\in\mathcal{D}(\Omega\times\Omega)$. In (26), each function is taken in its respective variable.

Now we get rid of the last term in the left-hand side of (26).

**Lemma 3.2.** *Assume the diffusion flux $a$ takes the form (2) with $S$ satisfying (7) and $a_0$ satisfying (4),(5). Then for all $\xi\in\mathcal{D}(\Omega\times\Omega)$, the limit $L$ of the expression*

$$\iint_Q\iint_Q\big(a(w,\nabla w)-a(\hat{w},\nabla\hat{w})\big)\cdot\big(\nabla w-\nabla\hat{w}\big)\big(H_\varepsilon^+\big)'\big(w(t,x)-\hat{w}(s,y)+\varepsilon\pi(x,y)\big)\,\xi$$

*as $\varepsilon\to 0$ is non-negative.*

Proof : The idea is the one of [26, Lemma 1]. We use chain rules, integrate by parts and exploit the continuity of $F$, the Lipschitz continuity of $S$ and the monotonicity of $a_0$. We write

$$a(w,\nabla w)-a(\hat{w},\nabla\hat{w})=\big(F(w)-F(\hat{w})\big) \\ +\big(S(w)-S(\hat{w})\big)a_0(\nabla w)+S(\hat{w})\big(a_0(\nabla w)-a_0(\nabla\hat{w})\big);$$

the scalar product of the latter term by $\big(\nabla w-\nabla\hat{w}\big)$ is nonnegative, by (4) and (7). Further, the support of the function $\big(H_\varepsilon^+\big)'\big(w(t,x)-\hat{w}(s,y)+\varepsilon\pi(x,y)\big)$ is included within the set $\big[|w(t,x)-\hat{w}(s,y)|<2\varepsilon\big]$; thus by the Lipschitz continuity of $S$ and the bound $0\leq \big(H^+_\varepsilon\big)'\leq \frac{1}{\varepsilon}$,

$$|S(w)-S(\hat{w})|\big(H^+_\varepsilon\big)'\big(w(t,x)-\hat{w}(s,y)+\varepsilon\pi(x,y)\big)\leq C\varepsilon\frac{1}{\varepsilon}1\!\!1_{[|w(t,x)-\hat{w}(s,y)|<2\varepsilon]}.$$

Now we see that the contribution of this term to $L$ is zero, because we have

$$(\nabla w-\nabla\hat{w})\cdot a_0(\nabla w)\,1\!\!1_{[|w(t,x)-\hat{w}(s,y)|\leq 2\varepsilon]}=G(t,s,x,y)1\!\!1_{[0<|w(t,x)-\hat{w}(s,y)|<2\varepsilon]}$$

for an $L^1$ function $G$ (recall that $\nabla w,\nabla\hat{w}\in L^p(Q)$ and $a_0(\nabla\hat{w})\in L^{p'}(Q)$, by assumption (5)), and the measure of the set $\big[0<|w(t,x)-\hat{w}(s,y)|<2\varepsilon\big]$ vanishes as $\varepsilon\to 0$.

It remains to treat the term

$$\iint_Q\iint_Q \big(F(w)-F(\hat{w})\big)\cdot\nabla w\,\big(H^+_\varepsilon\big)'\big(w(t,x)-\hat{w}(s,y)+\varepsilon\pi(x,y)\big)\,\xi \tag{27}$$

and the analogous term with $\nabla w$ replaced with $-\nabla\hat{w}$. Setting

$$\Psi_\varepsilon(w,\hat{w};\pi):=\int_{\hat{w}}^{w}(F(r)-F(\hat{w}))\big(H^+_\varepsilon\big)'\big(r-\hat{w}+\varepsilon\pi(x,y)\big)\,dr \tag{28}$$

and using the chain rule and integration-by-parts in variable $x$, we can rewrite the above term as

$$-\iint_Q\iint_Q \Psi_\varepsilon(w,\hat{w};\pi)\cdot\nabla_x\xi+\nabla_x\pi\cdot\big(F(\hat{w}+\varepsilon(1-\pi))-F(\hat{w}-\varepsilon\pi)\big)\,\xi. \tag{29}$$

Denoting by $\omega_{F,\hat{w}}$ the modulus of continuity of $F$ in a neighbourhood of $\hat{w}$, we have

$$0\leq|F(r)-F(\hat{w})|\,\big(H^+_\varepsilon\big)'\big(r-\hat{w}+\varepsilon\pi(x,y)\big)\leq\frac{1}{\varepsilon}\,\omega_{F,\hat{w}}(|w-\hat{w}|)\,1\!\!1_{[|w-\hat{w}|\leq 2\varepsilon]},$$

so that $\|\Psi_\varepsilon(w,\hat{w};\pi)\|_\infty\leq 2\omega_{F,\hat{w}}(2\varepsilon)$. Thus the first term in (29) is bounded by $2\omega_{F,hatw}(2\varepsilon)\,\|\xi\|_\infty$; further, the second term in (29) is bounded by $\|\nabla_x\pi\|_\infty\,\omega_{F,\hat{w}}(2\varepsilon)$. We conclude that (39) tends to zero as $\varepsilon\to 0$, using the dominated convergence theorem, the rough bound $\Psi_\varepsilon(w,\hat{w};\pi)\leq 2\max_{[\hat{w}-\varepsilon,\hat{w}+\varepsilon]}|F|$, the growth assumption (6) and the $L^p$ boundedness of $\hat{w}$. This ends the proof. $\diamond$

**Remark 3.3.** *If $\hat{w}$ is bounded, the end of the above proof becomes simpler (namely, we can take a uniform modulus of continuity $\omega_F$ on a compact containing the values of $\hat{w}\pm\varepsilon$). In [11], we show that the general case is reduced to this situation, using the idea of [37] (see Section 6).*

Using Lemma 3.2, we can drop the last term from the left-hand side of (26), replacing the equality sign with the inequality sign "$\leq$". Now we can proceed by approximation to extend the obtained inequality to a general measurable $[0,1]$-valued function $\pi$ on $\Omega\times\Omega$.

The next step would be to make $\pi$ converge to the function

$$p:(t,s;x,y)\mapsto \mathrm{sign}^+(j(v(t,x)))-j(\hat{v}(s,y))), \tag{30}$$

in order to benefit from (20). Here, we start using the assumption that $\hat{v}$ is a stationary solution, therefore we can drop the dependence on $s$. Because $\pi$ can only depend on $(x,y)$, we proceed by piecewise constant in $t$ approximation and we have delicate points to handle (see (33) below).

We start the argument with the following technical lemma.

**Lemma 3.4.** *There exists a family of partitions $0=t_0<t_1<\ldots<t_{N_m}=T$ (in the notation; we drop the dependence of the partition on $m\in\mathbb{N}$) such that*

*(i) for all $i$, $t_i$ are Lebesgue points of the map $j(v)$ considered as an $L^1(0,T)$ map with values in $L^1(\Omega)$;*

*(ii) the function $p$ in (30) is approximated in $L^1((0,T)\times\Omega^2)$ and a.e. by*

$$\pi^m(t;x,y):=\sum_{i=1}^{N_m}\pi_i(x,y)\,1\!\!1_{(t_{i-1},t_i]}(t),$$

*where $(\pi_i)_{i=1..N_m}$ are defined as $\pi_i(x,y):=\mathrm{sign}^+(j(v(t_i,x))-j(\hat{v}(y)))$.*

Proof : The non-Lebesgue points of $j(v)$ form a set of measure zero, whence (i) is easy to achieve. To get (ii), we set $p(t;x,y) = \mathrm{sign}^+(j(v(t_i,x)) - j(\hat v(y)))$ considered as an $L^\infty(0,T;L^1(\Omega^2))$ map; applying the Lusin theorem for $L^\infty(0,T;X)$ functions, $X = L^1(\Omega^2)$, for all $\varepsilon > 0$ we take a function $p_\varepsilon \in C([0,T];L^1(\Omega^2))$ such that $p \equiv p_\varepsilon$ on a set $(0,T)\setminus E^*_\varepsilon$, where $E^*_\varepsilon$ is of Lebesgue measure less than $\varepsilon$; moreover, by taking if necessary $\min\{p^+_\varepsilon, 1\}$, we can assume that $0 \le p_\varepsilon \le 1$.

Now take $N \in \mathbb{N}$ and take a uniform partition $(t^*_i)_{i=0}^N$ of $(0,T)$ with step $h = T/N$. Then we create $t_i$ as follows: $|t_i - t^*_i| \le h/4$, $t_i$ is a Lebesgue point of $j(v)$ and of $\mathrm{sign}^+(j(v(t_i)) - j(\hat v))$, and, whenever possible, $t_i \notin E^*_\varepsilon$. In particular, if $t_i \in E^*_\varepsilon$, then $E^*_\varepsilon$ contains the interval $[t^*_i - h/4, t^*_i + h/4]$ (up to a set of measure zero). Such intervals being disjoint, it is easily seen that the joint measure of all the intervals $I_i := (t_{i-1}, t_i]$ such that $t_i \in E^*_\varepsilon$ does not exceed $3\varepsilon$. We denote by $E_\varepsilon$ the union $E^*_\varepsilon \cup \big(\cup_{i\,:\,t_i\in E^*_\varepsilon} I_i\big)$; its measure does not exceed $4\varepsilon$.

Finally, taking $\pi^N$ according to the partition we've just created, we have

$$\|\pi^N - p\|_{L^1(0,T;L^1)} \le \|\pi^N - p_\varepsilon\|_{L^1((0,T)\setminus E_\varepsilon;L^1)} + \|\pi^N - p_\varepsilon\|_{L^1(E_\varepsilon;L^1)} + \|p_\varepsilon - p\|_{L^1(0,T;L^1)}.$$

Let $\omega_\varepsilon$ denote the modulus of continuity of $p_\varepsilon$ in $C([0,T];L^1(\Omega^2))$. By construction, the first term in the right-hand side is less than or equal to $\omega_\varepsilon(3h/2)$. The two other terms does not exceed $const\,\varepsilon$. Hence by taking $h = h(\varepsilon)$ small enough, we get $\|\pi^N - p\|_{L^1(0,T;L^1)} \le const\,\varepsilon$. Passing to a subsequence with $N = N_m$, we ensure the a.e. convergence of $\pi^m$ to $p$. $\diamond$

Now for all $i$, we combine (26) with Lemma 3.2 for test functions approximating $\xi 1\!\!1_{(t_{i-1},t_i)}(t)$ and with $\pi = \pi_i$. Thanks to Lemma 3.4(i) and because $J^{\sharp,\flat}(r_1, r_2;\pi)$ are continuous functions of $j(r_1)$, we get the following inequalities on each rectangle $(t_{i-1}, t_i) \times (0,T)$:

$$\begin{aligned}
(31)\qquad & -\int_0^T\!\!\int_\Omega\int_{t_{i-1}}^{t_i}\!\!\int_\Omega \Big[J^\sharp(v,\hat v;\pi_i)\,\xi_t + J^\flat(v,\hat v;\pi_i)\,\xi_s\Big] \\
&+ \int_0^T\!\!\int_\Omega\int_\Omega J^\sharp(v(t_i),\hat v;\pi_i)\,\xi(t_i,s) - \int_0^T\!\!\int_\Omega\int_\Omega J^\sharp(v(t_{i-1}),\hat v;\pi_i)\,\xi(t_{i-1},s) \\
&+ \int_0^T\!\!\int_\Omega\int_{t_{i-1}}^{t_i}\!\!\int_\Omega H(v,\hat v,\pi_i)\big(a(w,\nabla w) - a(\hat w,\nabla\hat w)\big)\cdot(\nabla_x + \nabla_y)\xi \\
&- \int_\Omega\int_{t_{i-1}}^{t_i}\!\!\int_\Omega J^\flat(v,\hat v_0;\pi_i)\,\xi(t,0) \le \int_0^T\!\!\int_\Omega\int_{t_{i-1}}^{t_i}\!\!\int_\Omega H(v,\hat v,\pi_i)\,(f - \hat f)
\end{aligned}$$

Now we piece together the inequalities (31), summing in $i$; we get

$$\begin{aligned}
(32)\qquad & -\iint_Q\iint_Q \Big[J^\sharp(v,\hat v;\pi^m)\,\xi_t + J^\flat(v,\hat v;\pi^m)\,\xi_s\Big] \\
&- \iint_Q\int_\Omega J^\sharp(v_0,\hat v;\pi^m(0^+))\,\xi(0,s) - \int_\Omega\iint_Q J^\flat(v,\hat v_0;\pi^m(t))\,\xi(t,0) \\
&+ \iint_Q\iint_Q H(v,\hat v,\pi^m)\big(a(w,\nabla w) - a(\hat w,\nabla\hat w)\big)\cdot(\nabla_x + \nabla_y)\xi \\
&+ R_m \le \iint_Q\iint_Q H(v,\hat v,\pi^m)\,(f - \hat f),
\end{aligned}$$

where the remainder term $R_m$ is nonnegative by construction of $\pi^m$:

$$(33)\qquad R_m = \sum_{i=1}^{N_m-1}\int_0^T\!\!\int_\Omega\int_\Omega \big(J^\sharp(v(t_i),\hat v;\pi_i) - J^\sharp(v(t_i),\hat v;\pi_{i+1})\big)\,\xi(t_{i-1},s) \ge 0.$$

Indeed, we recall (22) and the choice $\pi_i = \mathrm{sign}^+(j(v(t_i)) - j(\hat v))$; since $\xi$ is nonnegative, $R_m \ge 0$. Thus we can drop $R_m$ from (32).

Using properties (20),(21) of Lemma 3.1, with the help of the dominated convergence theorem we pass to the limit in (32) to get

$$
\begin{aligned}
(34)\quad & -\iint_Q\iint_Q (j(v)-j(\hat v))^+\,(\xi_t+\xi_s)\\
&\quad -\iint_Q\int_\Omega (j(v)-j(\hat v))^+\,\xi(0,s)-\int_\Omega\iint_Q (j(v)-j(\hat v))^+\,\xi(t,0)\\
&\quad +\iint_Q\iint_Q \operatorname{sign}^+(j(v)-j(\hat v))\,\big(a(w,\nabla w)-a(\hat w,\nabla\hat w)\big)\cdot(\nabla_x+\nabla_y)\xi\\
&\le \iint_Q\iint_Q \operatorname{sign}^+(j(v)-j(\hat v))\,(f-\hat f)\xi \le \iint_Q \Big[j(v)-j(\hat v),\,(f-\hat f)\,\xi\Big]^+_{L^1(Q)}
\end{aligned}
$$

(here for the second and third terms in the left-hand side, we have used the upper bound $J^{\sharp,\flat}(r_1,r_2;\pi)\le (j(r_1)-j(r_2))^+$ that is clear from the definition of $J^{\sharp,\flat}$). Note in passing that a.e. on $Q\times Q$, we have the equality

$$
H(v,\hat v,\pi^m)\,\big(a(w,\nabla w)-a(\hat w,\nabla\hat w)\big)=\operatorname{sign}^+(w-\hat w)\,\big(a(w,\nabla w)-a(\hat w,\nabla\hat w)\big).
$$

To conclude, we use the standard doubling of variables method of Kruzhkov [43], the upper semicontinuity of the brackets $\big[\,\cdot\,,\,\cdot\,\big]^\pm_{L^1(Q)}$ (see, e.g., [9] for the technique using brackets), and the following lemma inspired by an idea of Panov [54].

**Lemma 3.5.** *Assume that $v$ is a weak solution of (1) with initial datum (12). Then* $\operatorname*{ess\,lim}_{h\to 0^+}\|j(v)(h)-j_0\|_{L^1(\Omega)}=0$.

PROOF : The proof of the lemma is based upon the entropy inequalities, that are the Kato inequalities (19) with $\hat v\equiv k$, where $k$ is constant. In this case, the doubling of variables is avoided, and the arguments of the above proof (with the choice of $\pi=\pi(x)$ approximating $\operatorname{sign}^+(j(v(h,x))-j(k))$ ) with the rough bound $H(r_1,r_2;p)\le \operatorname{sign}^+(j(r_1)-j(r_2))$ ) yield

$$
\begin{aligned}
(35)\quad \int_\Omega (j(v(h))-j(k))^+\,\xi \le &\int_\Omega (j_0-j(k))^+\,\xi\\
&+\int_0^h\int_\Omega |f-\hat f|\xi+\int_0^h\int_\Omega |a(w,\nabla w)-a(k,0)|\,|\nabla\xi|
\end{aligned}
$$

for a.e $h\in(0,T)$, for all $\xi\in\mathcal D(\Omega)$, $\xi\ge 0$.

From (35) we deduce that $(j(v(h))-j_0)^+\to 0$ in $L^1(\Omega)$ as follows. For $\alpha>0$, we pick a finite family $(k_i)_i$ and a partition $(\Omega_i)_i$ of $\Omega$ such that

$$
\|j_0-\sum\nolimits_i j(k_i)1\!\!1_{\Omega_i}\|_{L^1(\Omega)}\le \alpha/9,
$$

with $\Omega_i$ obtained by intersecting $\Omega$ with the cells of a uniform cartesian grid of $\mathbb R^d$.

Replacing the family $(1\!\!1_{\Omega_i})_i$ by a partition of unity $(\xi_i)_i$ such that

$$
\|\sum\nolimits_i j(k_i)\,|1\!\!1_{\Omega_i}-\xi_i|\,\|_{L^1(\Omega)}\le\alpha/9,\quad \|\sum\nolimits_i j_0\,|1\!\!1_{\Omega_i}-\xi_i|\,\|_{L^1(\Omega)}\le\alpha/9,
$$

we use (35) with $k=k_i$ and $\xi=\xi_i$; we sum up in $i$. The outcome is

$$
(36)\quad \int_\Omega\sum_i (j(v(h))-j(k_i))^+\,\xi_i\le\int_\Omega\sum_i (j_0-j(k_i))^+\,\xi_i+\int_0^h\int_\Omega(|f-\hat f|+\mathcal F_\alpha)
$$

where $\mathcal F_\alpha=|a(w,\nabla w)-a(k_i,0)|\,|\nabla\xi_i|$ is the $L^{p'}(Q)$ function that only depends on $w$ and on the choice of $(k_i)_i$ and $(\xi_i)_i$. Now it is clear that the last term in (36) is smaller than $\alpha/3$ for $h < h_\alpha$. To conclude, note that

$$\begin{aligned}\int_\Omega (j(v(h))-j_0)^+ &= \int_\Omega (j(v(h))-j_0)^+ \sum_i \xi_i \\ &\le \int_\Omega \sum_i (j_0-j(k_i))^+ \xi_i + \int_\Omega \sum_i (j(v(h))-j(k_i))^+ \xi_i \\ &\le 3\frac{\alpha}{9} + \int_\Omega \sum_i (j_0 - j(k_i))^+ \xi_i + \frac{\alpha}{3} \le \alpha,\end{aligned}$$

due to the approximation properties behind the choice of $(k_i)_i$ and $(\xi_i)_i$.

Thus $(j(v(h))-j_0)^+$ goes to 0; the study of $(j(v(h))-j_0)^-$ is analogous. $\diamond$

Finally, notice that in order to get the Kato inequalities (19) for $\mathrm{sign}^-$, it is sufficient to exchange the roles of $v, \hat{v}$. We have shown the following result:

**Proposition 3.6.** *Consider problem (1),(12) with flux (2) under assumptions (3)-(6) and under the assumption (7) on $S$. The Kato inequalities (19) with $\xi = 0$ on $(0,T)\times\partial\Omega$ hold true if*

- *$v$ is a weak solution of the problem and*
- *$\hat{v}$ is a constant in time weak solution of the problem.*

To continue, it is necessary to bypass the restriction "$\xi = 0$ on $(0,T)\times\partial\Omega$" in the above result. In Section 3.3 and Section 4, we discuss two different ways for doing that. Namely, either one has to generalize the proof of inequalities (19) so that they allow for test functions $\xi$ non zero on $\partial\Omega$ (in which case one can put $\xi = 1\!\!1_{[0,t)}$ in (19), for a.e. $t \in (0,T)$); or one has to pass to the limit in (19) with a sequence $\xi_m \in \mathcal{D}([0,T)\times\Omega)$, $\xi_m \to 1$.

3.2. **Doubling of variables inside the domain: a variant.** In [27], Carrillo and Wittbold obtained Kato inequalities (for renormalized solutions) for (1),(12) with $\varphi = Id$ under the following additional assumption on a Leray-Lions kind convection-diffusion flux $a$:

$$(a(r,\xi)-a(\rho,\eta))\cdot(\xi-\eta)+C(r,s)(1+|\xi|^p+|\eta|^p)|r-s| \ge \Gamma(r,s)\cdot\xi+\hat{\Gamma}(r,s)\cdot\eta \tag{37}$$

where $\Gamma, \hat{\Gamma} : \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}^d$ and $C : \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ are continuous.

The flux (2) with $S \equiv 1$ is a particular case where (37) is satisfied. In Section 3.1 above, we have prepared the ground for uniqueness results for fluxes (2) with nonnegative bounded Lipschitz continuous $S$.

In this section, we give another modification of the doubling of variables argument suitable for fluxes (2) with merely continuous $S$ satisfying (8). Yet notice that, whenever $a_0$ is linear (or, more generally, homogeneous of degree $p$), the term $S(w)a_0(\nabla\varphi(v))$ can be rewritten as $a_0(\nabla\varphi_S(v))$ for a suitable continuous non-decreasing function $\varphi_S$; thus $S \equiv 1$ remains the most interesting case, and for the time being, the below refinement of the techniques lacks true applications.

**Proposition 3.7.** *Consider problem (1),(12) with flux (2) under assumptions (3)-(6) and under the assumption (8) on $S$. The Kato inequalities (19) with $\xi = 0$ on $(0,T)\times\partial\Omega$ hold true if*

- *$v$ is a weak solution of the problem and*
- *$\hat{v}$ is a constant in time weak solution of the problem.*

PROOF (SKETCHED): The only point different from the proof of Proposition 3.6 is that Lemma 3.2 should be replaced. For the sake of simplicity, assume that $\varphi$ is strictly increasing, so that we can drop the term $\varepsilon\pi$ from the calculations; also assume that $\hat{w}$ is bounded (see Remark 3.3). We have to show that the limit $L$, as $\varepsilon \to 0$, of the term

$$\iint_Q \iint_Q \big(a(w,\nabla w)-a(\hat{w},\nabla\hat{w})\big)\cdot\big(\nabla w - \nabla\hat{w}\big)\ \big(H_\varepsilon^+\big)'\big(w(t,x)-\hat{w}(s,y)\big)\ \xi,$$

is nonnegative. Introducing the function $G := F/S$, we have

$$
\begin{aligned}
(38)\quad & \big(a(w,\nabla w) - a(\hat w,\nabla \hat w)\big)\cdot\big(\nabla w - \nabla \hat w\big)\\
&= \Big(S(w)\big(G(w)+a_0(\nabla w)\big) - S(\hat w)\big(G(\hat w)+a_0(\nabla \hat w)\big)\Big)\cdot\big(\nabla w - \nabla \hat w\big)\\
&= S(w)\big(a_0(\nabla w) - a_0(\nabla \hat w)\big)\cdot\big(\nabla w - \nabla \hat w\big)\\
&\quad + S(w)\big(G(w)-G(\hat w)\big)\cdot\nabla w - S(w)\big(G(w)-G(\hat w)\big)\cdot\nabla \hat w\\
&\quad + \big(S(w)-S(\hat w)\big)\big(G(\hat w)+a_0(\nabla \hat w)\big)\cdot\nabla w - \big(S(w)-S(\hat w)\big)\big(G(\hat w)+a_0(\nabla \hat w)\big)\cdot\nabla \hat w.
\end{aligned}
$$

The first term in the right-hand side of (38) is nonnegative. The contribution to $L$ of the second term is treated as in Lemma 3.2, using the function

$$\Psi^1_\varepsilon(w,\hat w) := \int_{\hat w}^{w} (G(r)-G(\hat w))S(r)\big(H_\varepsilon^+\big)'\big(r-\hat w\big)\,dr.$$

in the place of the function (28). Similarly, for the third term, we use

$$\Psi^2_\varepsilon(w,\hat w) := \int_{\hat w}^{w} (G(w)-G(r))\big(H_\varepsilon^+\big)'\big(w-r\big)\,dr;$$

we rewrite this term under the form

$$
(39)\qquad \iint_Q S(w)\iint_Q \operatorname{div}_y \Psi^2_\varepsilon(w,\hat w)\,\xi = -\iint_Q S(w)\iint_Q \Psi^2_\varepsilon(w,\hat w)\cdot\nabla_y\xi,
$$

and conclude using the fact that $\|\Psi^2_\varepsilon\|_\infty$ vanishes as $\varepsilon\to 0$. The fourth term is treated in the same way as the third one; here the $y$-dependent term $\big(G(\hat w) + a_0(\nabla \hat w)\big)$ plays the role of $S(w)$ in the calculation (39), and the integration by parts is in $x$. Finally, the last term in (38) gives rise to the following contribution:

$$\iint_Q\iint_Q \big(H_\varepsilon^+\big)'\big(w-\hat w\big)\,\frac{S(w)-S(\hat w)}{S(\hat w)}\nabla\hat w\cdot\Big(\xi\; S(\hat w)\big(G(\hat w)+a_0(\nabla\hat w)\big)\Big).$$

Here we notice that since $\hat v$ does not depend on time,

$$\operatorname{div}\big[S(\hat w)\big(G(\hat w)+a_0(\nabla \hat w)\big)\big] = \operatorname{div} a(\hat w,\nabla\hat w) = f\in L^1(Q).$$

Thus we can integrate by parts in variable $y$ with the auxiliary function

$$\Psi^3_\varepsilon(w,\hat w) := \int_{\hat w}^{w} \frac{S(w)-S(r)}{S(r)}\big(H_\varepsilon^+\big)'\big(w-r\big)\,dr;$$

because $S$ is continuous and lower bounded, we have $\|\Psi^3_\varepsilon\|_\infty\to 0$ as $\varepsilon\to 0$. $\diamond$

With the help of Proposition 3.7, of Remark 3.8 below, using the notion of integral solution (see Definition 2.5 and Section 5), one can establish uniqueness results for (1),(2) with merely continuous $S$ bounded from above and from below.

3.3. **Doubling of variables up to the boundary.** Taking into account the boundary conditions within the doubling of variables procedure is a hard task. For the homogeneous Dirichlet boundary conditions, this has been achieved by Carrillo in [26]. For non-homogeneous Dirichlet boundary conditions satisfying rather strong regularity assumptions, this was done in [47, 48, 59] and in [4, 2, 3]. For the Neumann boundary conditions, a specific procedure was designed in [9].

Notice that in each case, one has to establish entropy inequalities of the kind (16) with test functions non necessarily zero on the boundary; these inequalities usually contain boundary terms. Then the doubling of variables procedure yields boundary terms that are non-negative and can be dropped. In the next paragraphs, we briefly recall the arguments used in the aforementioned proofs.

3.3.1. *The Dirichlet BC case.* For $g \equiv 0$, Carrillo [26] gets entropy inequalities for equation (1) only for a restricted choice of couples $(k, \xi)$. Namely, the test function $\xi$ in (16) is allowed to be nonzero at the boundary only for $k \geq 0$ (in the "sign$^+$" inequalities) or for $k \leq 0$ (in the "sign$^-$" inequalities). In the doubling of variables procedure, the positive and negative parts of the two solutions are separated and treated apart, using entropy inequalities (16) for the aforementioned couples $(k, \xi)$ (see, e.g., [7, Lemma A.2] for the elementary calculation underlying this separation). The argument is lengthy; we refer to the original paper [26] and to [7, Lemma A.5] where the different steps of the proof "near the boundary" are highlighted.

Notice that although the result of [26] was stated for the linear diffusion (i.e, $a_0 = Id$) and under the additional assumption that $\varphi$ is strictly increasing at zero, the linearity of $a_0$ was not essential in the arguments (see [27]). Later, a hint suppressing the assumption $\varphi^{-1}(0) = 0$ of [26] was designed by the authors in [10].

**Remark 3.8.** *Let us mention that this technique of* [26] *for the homogeneous problem works for the convection-diffusion fluxes* (2) *under the assumptions of Proposition 3.6. One can follow, e.g., the arguments of* [7, Lemma A.5] *with the calculations of Proposition 3.6 in hand, in order to treat the neighbourhood of the boundary.*

Although the separation argument of Carrillo is not appropriate for non-constant boundary conditions, it is feasible to use the idea locally, near each point of the boundary where the Dirichlet condition $g$ is continuous; such technique was developed by Ammar, Carrillo and Wittbold [4] in the context of a pure hyperbolic nonlinear convection problem. These techniques were extended by Ammar [2, 3] to the triply nonlinear framework. Also notice that piecewise constant Dirichlet boundary conditions can be treated as in [26], at least for the case of linear diffusion $a_0 = Id$ (cf. [12] where this argument is used to combine Dirichlet and Neumann BC). Indeed, we can proceed by a simple partition of unity, making test functions $\xi_h$ vanish only in an $h$-neighbourhood of the discontinuities of $g$ on $\partial\Omega$ (the set of discontinuities has zero capacity, hence the terms with $\nabla\xi_h$ are easy to control). One can hope to treat the case of piecewise continuous Dirichlet datum $g$ by combining this idea with the techniques of [4, 2, 3].

The next key idea to treat non-homogeneous boundary conditions was inspired by the work [50] of Otto on conservation laws. The point is to get up-to-the-boundary entropy inequalities for every couple $(k, \xi)$; the price to pay is the presence of a "remainder term" coming from the boundary. For the Carrillo choices of $(k, \xi)$, this term was (formally) non-negative and therefore it was dropped (see Rouvre and Gagneux [57]; cf. [47, Remark 1.2]). For general $(k, \xi)$, even the definition of such remainder term is not straightforward; the theory of weak boundary traces for divergence-measure fields (see [28] and the previous work by Anzellotti) can be used to make them meaningful. Typical tools are [47, Definition 1.1, Lemma 2.2] and [59, Lemma 1] that are used to "generate" boundary terms from sequences of test functions $(\xi_h)_h$ with gradient concentrated at an $h$-neighbourhood of the boundary. This approach is used in the works Mascia, Porretta, Terracina [47], Michel, Vovelle [48] and Vallet [59], the latter work presenting most general results for hyperbolic-parabolic problems with $(t, x)$-dependent coefficients. The context of these works is much more general than the ours, because it includes hyperbolic degeneracy; yet the application of these arguments to (1) remains lengthy. Moreover, only linear diffusion corresponding to $a_0 = Id$ (thus to $-\operatorname{div} a_0(\nabla w) = -\Delta w$) is allowed.

A simpler technique for treating the non-homogeneous Dirichlet problem for (1) is discussed in Section 4; it is not based on up-to-the-boundary entropy inequalities, but upon extension to the boundary of the local Kato inequalities (19) that were already proved.

3.3.2. *The Neumann BC case.* For the case of Neumann boundary conditions, at least those that are regular enough, there is no difficulty in writing down up-to-the-boundary entropy inequalities. Yet the attempts to use them within the doubling of variables procedure run into major problems, except for the case where the solutions are so regular that the Neumann condition (10) is assumed in the a.e. sense (more precisely, as the strong $L^1$ normal trace of $a(w,\nabla w)$ on $(0,T)\times\partial\Omega$ ).

In practice, we do not know how to ensure this regularity unless solutions are of the class $C^1$ up to the boundary. Such regularity (more precisely, Hölder $C^{1,\alpha}$ regularity) is well known for quasilinear or nonlinear stationary problem (17) with $L^\infty$ source term $h$ and appropriate Hölder regular Neumann datum $s$ and boundary $\partial\Omega$; we refer in particular to Lieberman [44] and references therein. Analogous regularity results for the evolution problem (1) exist in the literature but they are much more difficult to apply. Thus, the only easy task is to get uniqueness for the stationary problem (17) with regular Neumann datum (10) and $L^\infty$ source $h$.

At this point, the idea of the work of Andreianov and Bouhsiss [9] was to break the symmetry in the application of the doubling of variables method, by taking test functions that are zero on the boundary $Q\times\big((0,T)\times\partial\Omega\big)$ of $Q\times Q$ but non-zero on the boundary $\big((0,T)\times\partial\Omega\big)\times Q$. As it is demonstrated in [9], in this case we can assume that only one solution is $C^1$ up to the boundary, and the other solution can be arbitrary. Hence we have the following statement similar to Proposition 3.7:

**Proposition 3.9.** *(see [9]) Assume $\Omega$ is a bounded $C^2$ domain of $\mathbb{R}^d$; assume $\varphi=Id$, $a_0=Id$, $S\equiv 1$ and assume $F$ is a locally $C^{0,\alpha}$ Hölder continuous function, $\alpha>0$, with at most linear growth of $F$ at infinity.*

*The Kato inequalities (19) with $\xi$ not necessarily zero on $(0,T)\times\partial\Omega$ hold true if*

- *$v$ is a weak solution of the evolution problem (1),(2) with homogeneous Neumann boundary condition (10);*
- *$\hat v$ is a weak solution of the stationary problem (17),(2) with homogeneous Neumann BC (10) and with source term $h\equiv\hat f$ in $L^\infty(\Omega)$.*

Extension of this result to non-homogeneous or mixed boundary conditions and nonlinear diffusions $a_0$ is the subject of the work [12] of Soma and the authors.

Clearly, it is enough to take $\xi\equiv 1$ in the Kato inequalities stated in Proposition 3.9 in order to deduce inequalities (18) of Definition 2.5. In this way, we can justify that weak solutions of the evolution problem treated in Proposition 3.9 are integral solutions of the associated abstract evolution problem. In Section 5, we show that this kind of result readily yields uniqueness of weak solutions.

## 4. Kato inequalities: "going to the boundary"

In this section, we assume that either $\Omega=\mathbb{R}^d$ or $\Omega$ is bounded and a non-homogeneous Dirichlet boundary condition (9) is prescribed on $(0,T)\times\partial\Omega$. The starting point is the *local* Kato inequalities (19), i.e. Kato inequalities with $\xi\in\mathcal{D}([0,T)\times\Omega)$. The goal is to pass to the limit with some sequence $(\xi_h)_h$ converging to 1 on $(0,T)\times\Omega$.

Let us stress that there are at least two strategies in choosing such sequences $(\xi_h)_h$. The first one is to construct $\xi_h$ more or less explicitly, using only the geometry of the domain (this is the case in [48, 59] and also in [23, 46, 45] described below). The second one is to construct $(\xi_h)_h$ by solving a PDE related to some of the terms in (19) (this was the case in [47]); this is a Holmgren-type approach, and it may lead to finer constructions.

4.1. **Cauchy problem in the whole space.** In the case where $\Omega$ is the whole space, one has no choice but to start with the local Kato inequalities (19). The ground was prepared by the works on uniqueness of entropy solutions for conservation laws with non-Lipschitz flux $F$; this includes the results of Kruzhkov, Hil'debrand, Panov, Bénilan, Andreianov (see in particular Bénilan and Kruzhkov [23]; other references can be found in [46, 45, 13] and further works by Panov). Then Maliki and Touré in [46] adapted the technique of Bénilan and Kruzhkov [23] to the context of the hyperbolic-parabolic problem $u_t-\operatorname{div}F(u)+\Delta\varphi(u)=0$. The linearity of $a_0=Id$ is essential in this argument, and restrictions on the modulus of continuity of $F$ (those known from the work [23]) and new restrictions on the modulus of continuity of $\varphi$ are needed, except in low dimension.

In [13], Andreianov and Maliki constructed a new family of test functions by truncating the fundamental solution of the Laplace operator (the restriction $a_0 = Id$ remains essential), and managed to remove the restrictions on $\varphi$. The result applies to bounded entropy solutions of $u_t - \operatorname{div} F(u) + \Delta\varphi(u) = 0$. Here we point out that the proof of [13] works also for the case of nonlinear $j$, thus we deduce uniqueness result for bounded weak solutions in the whole space of problem (1),(2) with $a_0 = Id$.

4.2. **The non-homogeneous Dirichlet problem.** Here we describe the technique developed by the authors in [11]. We need the linearity assumption on $a_0$; consider the case $a_0 = Id$ (thus we can always take $S \equiv 1$) in (2).

For $h > 0$, define $\Omega_h := \{x \in \Omega \,|\, \operatorname{dist}(x, \partial\Omega) < h\}$, $\xi_h^0 := \min\{1, \frac{1}{h}\operatorname{dist}(x, \partial\Omega)\}$. The family of distance-to-the-boundary functions $(\xi_h^0)_h$ converges to 1 a.e. on $\Omega$ as $h \to 0$, in fact this is the simplest candidate for testing the Kato inequalities (19). Yet it is not easy to analyze the sign of the weak trace boundary term generated as

$$
\begin{aligned}
(40) \quad \limsup_{h\to 0} \iint_Q & \operatorname{sign}^+(w - \hat{w})\big(a(w,\nabla w) - a(\hat{w},\nabla\hat{w})\big)\cdot\nabla\xi_h^0 \\
&\equiv \limsup_{h\to 0} \iint_Q \Big(\operatorname{sign}^+(w-\hat{w})(F(w) - F(\hat{w})) + \nabla(w-\hat{w})^+\Big)\cdot\nabla\xi_h^0
\end{aligned}
$$

(in the above transformation, we used the expression $a(w,\nabla w) = F(w) + \nabla w$ and the chain rule for $\nabla(w-\hat{w})^+$). Our choice is to adapt $\xi_h$ not only to the geometry of $\partial\Omega$, but also to the inequality (19) on which the test function will be used (cf. the construction in [47]). First consider $u_h$ the solution of the auxiliary problem

$$
(41) \qquad -\Delta u_h = 0 \ \text{ in } \Omega_h, \ \ u_h - \xi_h^0 \in H_0^1(\Omega_h).
$$

Then we set $\xi_h := 2\min\{u_h, 1/2\}$. By a classical result, $\xi_h$ is a super-solution of the same problem (41), in particular, we have (upon necessary justifications)

$$
\iint_Q \nabla(w-\hat{w})^+\cdot\nabla\xi_h = \iint_Q \big(-\Delta\xi_h\big)\,(w-\hat{w})^+ \geq 0.
$$

Now, assuming, e.g., that $\Omega$ is a weakly Lipschitz domain, we have the uniform in $h$ bound $\operatorname{meas}(\Omega_h) \leq Ch$ for some $C > 0$, and thus

$$
\frac{1}{C} \leq \int_\Omega |\nabla\xi_h^0| \quad \text{and} \quad \int_\Omega |\nabla\xi_h^0|^2 \leq \frac{C}{h^2}.
$$

The same bounds for $|\nabla\xi_h|$ are derived from the Poincaré-Friedrichs inequality and from the variational interpretation of the auxiliary problem (41). Then we can conclude that the limit (40) (with $\xi_h^0$ replaced by $\xi_h$) its nonnegative, provided that

$$
I_h := \iint_Q \operatorname{sign}^+(w-\hat{w})|F(w) - F(\hat{w})|\,|\nabla\xi_h| \longrightarrow 0 \ \text{ as } h \to 0.
$$

Using a concave modulus of continuity $\omega_F$ of $F$ and a weighted Jensen inequality, we get

$$
I_h \leq \big(\textstyle\iint_Q|\nabla\xi_h|\big)\,\omega_F\Big(\big(\textstyle\iint_Q|\nabla\xi_h|\big)^{-1}\displaystyle\iint_Q (w-\hat{w})^+|\nabla\xi_h|\Big).
$$

Then we show that the right-hand side of the above inequality vanishes as $h \to 0$, thanks to the Cauchy-Schwarz inequality and to the Poincaré-Friedrichs inequality (notice that $(w-\hat{w})^+$ is zero on $\partial\Omega$), namely

$$
\frac{1}{h^2}\int_{\Omega_h} |(w-\hat{w})^+|^2 \leq C\int_{\Omega_h} |\nabla(w-\hat{w})^+|^2 \to 0 \ \text{ as } h \to 0.
$$

This concludes the argument; now we can take $\xi \equiv 1$ in space (in time, we take $\xi = 1\!\!1_{[0,T)}$ by approximation). From Proposition 3.6 we derive

**Proposition 4.1.** *(cf. [11]) Consider problem (1),(2) with $a_0 = Id$ and non-homogeneous time-independent Dirichlet boundary condition (9).*

*The Kato inequalities (19) with $\xi$ not necessarily zero on $(0,T)\times\partial\Omega$ hold true if*

- *$v$ is a weak solution of the problem and*
- *$\hat{v}$ is a constant in time weak solution of the problem.*

In [11], we give the analogous result for solutions of the stationary problem (17) with $a_0$ close to linear. This slight improvement makes apparent the idea behind the construction of the test functions $\xi_h$ in the works [11] and [13]: namely, $\xi_h$ solves a kind of adjoint PDE defined according to the Kato inequality.

To give a simple (and very restrictive) example, assume that $j = \varphi = Id$ and the jacobian $Da_0$ of $a_0$ is a symmetric bounded matrix with $Da_0(\xi)\eta \cdot \eta \geq \frac{1}{C}|\eta|^2$. Then the adjoint problem associated with (19) is the backward problem

$$(u_h)_t + \operatorname{div} P(t,x)\nabla u_h = 0, \quad u_h|_{t=T} = \xi_h^0, \quad u_h(t) - \xi_h^0 \in H_0^1(\Omega_h) \text{ for a.e. } t \in (0,T) \tag{42}$$

with the matrix $P$ defined from $w, \hat{w}$ by $P := \int_0^1 Da_0\big(\theta\nabla w + (1-\theta)\nabla\hat{w}\big)\, d\theta$. In this case, the solution $u_h$ of (42) replaces the solution of (41) for the construction of $\xi_h$.

In any more general situation (e.g., for $a_0$ corresponding to $p \neq 2$) the associated adjoint problem is of a singular or degenerate type; thus the method of [11] runs into major difficulties.

4.3. **The Neumann problem.** It appears that the strategy of this section cannot apply for the Neumann boundary conditions, unless one shows existence of strong boundary traces for $a(w, \nabla w)$. Surprisingly, strong trace results now appear as generic for the case of pure conservation laws (see in particular Panov [55]); but there is little hope to justify that the terms of the kind $a_0(\nabla w)$ admit strong normal traces, except for the stationary problem (17) in space dimension one.

4.4. **Conclusions.** The strategy of Section 3.3 and the strategy adapted in this section can be seen as concurrent, or complementary. Notice that in Section 3.3, the PDE is used up to the boundary; and in this section, in a small neighbourhood of the boundary we "forget" the precise information coming from the PDE and use only the information on the spaces to which the solutions belong.

For the non-homogeneous Dirichlet problem the approach of this section remains restricted to linear diffusions $a_0$. Yet it is by far less demanding than the one of [47, 48, 59] (also restricted to linear $a_0$) discussed in Section 3.3. The technique of Ammar and al. ([4, 2, 3]) mentioned in Section 3.3 is also heavy but it offers an alternative for treating both nonhomogeneous Dirichlet conditions and nonlinear diffusions $a_0$. Both techniques of [47, 48, 59] and of [4, 2, 3] were designed for hyperbolic-parabolic problems, much more general and difficult then problem (1). Presently, the technique of [11] is limited to the framework of Stefan-type problems (1), but it is feasible to combine the argument with the strong trace technique for quasi-solutions of conservation laws (see Panov [55]). Such generalization is an open problem.

Further, for the Neumann problem, the approach of Section 3.3 seems to be the only one that provides rather general results.

On the contrary, for the Cauchy problem in the whole space only the approach of this section applies, for linear diffusions $a_0$. Let us stress that little is known on the uniqueness of weak solutions in the whole space for convection-diffusion problems with nonlinear $a_0$, especially when $p > 2$.

## 5. Use of integral solutions and of partial comparison arguments

On two occasions, in Proposition 3.7 and Proposition 3.9, we found out that the doubling of variables procedure may require some regularity of the solutions. Breaking the symmetry of the classical Kruzhkov doubling argument allowed us to impose such regularity restrictions only on one of the two solutions $v, \hat{v}$ (a similar reasoning is given in [4], where a general solution with $L^\infty$ Dirichlet datum is compared to a "regular" solution with a continuous Dirichlet datum).

Regularity for the stationary equation (17) being a simpler issue than the regularity for the evolution equation (1), we were led to compare a solution $v$ of (1) to a "regular" solution $\hat{v}$ of the associated stationary problem (17). To be specific, in the framework of Proposition 3.9 "regularity of $\hat{v}$" means that $\hat{w} = \varphi(\hat{v}) \in C^1(\overline{\Omega})$. In the context of Proposition 3.7 "regularity of $\hat{v}$" means that $\operatorname{div} a(\hat{w}, \nabla \hat{w}) \in L^1(\Omega)$.

Also in Proposition 4.1 we have the same situation: a solution to the evolution problem is compared to a stationary solution (no additional regularity is required on this occasion). This time, the simplification lies in the fact that the doubling of the time variable is unnecessary, and we get a simpler proof than the one of [24].

Let us point out how to convert Propositions 3.7, 3.9, 4.1 into uniqueness results for the respective evolution problems. We use the tools of nonlinear semigroups governed by accretive operators on the space $L^1(\Omega)$. We refer to [20, 22] for the background and definitions of the terms used in this section.

First, we can apply Propositions 3.7, 3.9, 4.1 to two "regular" solutions of the stationary problem (17) with the corresponding boundary condition; it is crucial that the boundary condition is independent of $t$. We get the $L^1$ contraction property for such solutions. Then we define the operator $A$ on $L^1(\Omega)$ associated with "regular" solutions of (17) by its graph (roughly speaking, through the relation $(I + A)\hat{u} = \hat{f} + \hat{u}$ ):

$$(\hat{u}, \hat{f}) \in A \;\text{ iff }\; \hat{u} = j(\hat{v}),\ \hat{v} \text{ being a "regular" solution of (17) with } h = \hat{f} + \hat{u};$$

the contraction property implies that $A$ is an accretive operator on $L^1(\Omega)$. We need an existence analysis for such "regular" solutions of (17) in order to establish that the closure $\overline{A}$ of $A$ is an $m$-accretive operator. To be specific, in the framework of Proposition 3.9 we have existence of such "regular" solutions for (17) provided $h \in L^\infty(\Omega)$ (see [9] and [44]). Because $L^\infty(\Omega)$ is dense in $L^1(\Omega)$, the corresponding operator $\overline{A}$ is indeed $m$-accretive. In the framework of Propositions 3.7, 4.1, existence of a weak solution for the stationary problem e.g. with source $h \in L^\infty(\Omega)$ can be obtained by approximation, and the "regularity" of this solution is automatic.

The $m$-accretivity implies that for all initial datum $j_0$ in the domain $\overline{D(A)}$ of $\overline{A}$ and for all $f \in L^1(Q)$ there exists a unique mild solution of the abstract evolution problem $u_t + Au = f$ on $(0,T)$, $u(0) = j_0$. It remains to characterize the closure of the domain of $A$, which is a standard task in applications of the nonlinear semigroup theory; in most of the cases, one manages to show that $D(A)$ is dense in $L^1(\Omega, \overline{j(\mathbb{R})})$ (see, e.g., [9, 14, 15]). Then the so constructed mild solution is also the unique integral solution of our abstract evolution problem with initial datum $j_0$, see [20, 22, 17, 21]. We remind, in passing, the constraints (12),(11) on $j_0$ and $j(\cdot)$.

And now, the Kato inequalities of Propositions 3.7, 3.9, 4.1 (with $\xi \equiv 1$) exactly mean that every weak solution $v$ to the evolution problem (1),(12) (with the same BC as for (17)) corresponds to $u = j(v)$ which is an integral solution of the associated abstract evolution problem $u_t + Au = f$. In particular, the fact that $u(t) - j_0 \to 0$ as $t \to 0$ (also shown in Lemma 3.5) easily follows from the Kato inequalities and from the density of $D(A)$ (more generally, the time continuity of $u$ from the right is shown in this way). We conclude to the uniqueness of $j(v)$ such that $v$ is a weak solution to (1).

## 6. Renormalized solutions: a hint for uniqueness

In the work of Igbida, Sbihi and Wittbold [37] (see also [11]), the question of uniqueness of a renormalized solution to (1) with $a(r,\xi) = \xi + F(r)$ (i.e., with $a_0 = Id$) was reduced to the $L^1$ contraction principle for weak solutions for an auxiliary equation. This is quite natural, in view of the meaning of the renormalized formulation. Indeed, for $a_0 = Id$ Definition 2.3(i) can be seen as the weak formulation for the problem

$$j_S(v)_t - \mathrm{div}\,(S(w)F(w) + S(w)\nabla w) = f_S, \quad j_S(r) := \int_0^r S(\varphi(z))\,dj(z), \tag{43}$$

with $f_S := S(w)f - S'(w)\,a(w,\nabla w)\cdot\nabla w \in L^1(Q)$. Notice that if $F\equiv 0$ (the general case is subtler, see (45),(46) below), the constraint (ii) of Definition 2.3 makes $f_S$ converge to $f$ in $L^1(Q)$ as the renormalization function $S$ goes to 1.

Thus taking $S\geq 0$ and setting $\varphi_S(r) := \int_0^{\varphi(r)} S(z)\,dz$, observing that $j_S,\varphi_S$ are continuous non-decreasing functions and that $S(w)F(w) = F_S(\varphi_S(v))$ for some continuous and bounded function $F_S$, we see that (43) can be recast as

$$j_S(v)_t - \mathrm{div}\,(F_S(w_S) + \nabla w_S) = f_S, \quad w_S = \varphi_S(v) \;\text{ with }\; j_S(v)|_{t=0} = j_S(v_0). \tag{44}$$

Moreover, $S$ being compactly supported in $\mathbb{R}$, we have $w_S \in L^2(0,T;H^1(\Omega))$, so that a renormalized solution $v$ is also the weak solution for the whole family of formulations (44) with $S\geq 0$.

Two renormalized solutions $v,\hat v$ of (1) are weak solutions of the same auxiliary equation with the source terms $f_S = S(w)f - S'(w)a(w,\nabla w)\cdot\nabla w$ and $\hat f_S := S(\hat w)\hat f - S'(\hat w)a(\hat w,\nabla\hat w)\cdot\nabla\hat w$, respectively. Whenever (44) falls in the scope of problems for which the $L^1$ contraction principle is known (this is the case, e.g., for the homogeneous Dirichlet boundary condition), we write down the contraction principle (i.e., the Kato inequality with the test function $\xi$ going to $1\!\!1_{[0,t]}$)

$$\|j_S(v)(t)-j_S(\hat v)(t)\|_{L^1(\Omega)} \leq \|j_S(v_0)-j_S(\hat v_0)\|_{L^1(\Omega)} + \int_0^t \Big[j(v)-j(\hat v),(f_S-\hat f_S)\Big]_{L^1(\Omega)}$$

and pass to the limit as $S\longrightarrow 1$ on $\mathbb{R}$ using, e.g., $S_M(z) = \min\{1,(M-|z|)^+\}$. Recall that

$$\begin{aligned} f_S-\hat f_S = \big(S(w)f - S(\hat w)\hat f\big) + S'(w)\,a_0(\nabla w)\cdot\nabla w - S'(\hat w)\,a_0(\nabla\hat w)\cdot\nabla\hat w \\ + \big(F(w)\cdot\nabla S(w) - F(\hat w)\cdot\nabla S(\hat w)\big). \end{aligned} \tag{45}$$

For the last term in (45), the following argument should be used (it also applies in the more general context of [37, claim (3.5)]):

$$\begin{aligned} &\Big[j(v)-j(\hat v), \big(F(w)\cdot\nabla S(w) - F(\hat w)\cdot\nabla S(\hat w)\big)\Big]_{L^1(\Omega))} \\ &\qquad = \int_\Omega \mathrm{sign}\,(w-\hat w)\big(F(w)\cdot\nabla S(w) - F(\hat w)\cdot\nabla S(\hat w)\big) \\ &= \int_\Omega \mathrm{div}\,\Big(\int_{\min\{w,\hat w\}}^{\max\{w,\hat w\}} F(s)S'(s)\,ds\Big) = \int_{\partial\Omega}\Big(\int_{\min\{g,\hat g\}}^{\max\{g,\hat g\}} F(s)S'(s)\,ds\Big)\cdot n = 0, \end{aligned} \tag{46}$$

because the boundary conditons $g,\hat g$ coincide. The remaining terms in (45) converge to $f-\hat f$ strongly in $L^1(Q)$ as $S$ goes to 1 on $\mathbb{R}$ (due, in particular, to Definition 2.3(ii)). Finally, $j_S$ converges to $j$ on $\mathbb{R}$, so that at the limit $S\longrightarrow 1$ we get the $L^1$ contraction property for renormalized solutions of (1).

This proof is much simpler than the customary direct proofs of uniqueness of a renormalized solution. The reduction argument of [37] carries on to the case of a homogeneous of degree $p$ nonlinearity $a_0$ (this includes the celebrated $p$-laplacian diffusions); but in general, the form $a(z,\xi) = F(z) + a_0(\xi)$ of the flux considered in most of the papers on the subject does not allow for such reduction. It was the purpose of Section 3 to extend the doubling of variables technique to the diffusions of the form $a(z,\xi) = F(z) + S(z)a_0(\xi)$ with Lipschitz non-negative nonlinearity $S$. Now with the help of Remark 3.8, we readily extend the uniqueness approach of [37] to renormalized solutions of the homogeneous Dirichlet problem (1),(2) with general, not necessarily homogeneous, Leray-Lions diffusion flux $a_0$.

Notice that, e.g., for a vanishing at infinity flux $F$, with the reduction argument of Igbida, Sbihi and Wittbold [37] one readily extends to the framework of renormalized solutions the results of [9] on the homogeneous Neumann problem.

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